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Article

Discovery of the heart in mathematics: Modeling the chaotic behaviors of quantized periods in the Mandelbrot Set

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SUMMARY

The phenomenon of small changes leading to drastic effects is fundamental to Chaos Theory. Understanding the chaos in our world could provide more control over the systems that govern the universe. Therefore, this study aimed to predict and explain chaotic behavior in the Mandelbrot Set, one of the world's most popular models of fractals and exhibitors of Chaos Theory. We hypothesized that repeatedly iterating the Mandelbrot Set's characteristic function would give rise to a more intricate layout of the fractal and elliptical models that predict and highlight "hotspots" of chaos through their overlaps. While a novel method of discovering miniature versions of the Mandelbrot fractal was discovered and a statistically significant transformation function was developed, overlaps of the elliptical models were not supported to exhibit higher levels of chaos. Many biological and natural phenomena such as the heartbeat, lung vessels, neurons, weather, the stock market, and more, are both chaotic in nature and can be described using fractal-based models. The positive and negative results from this study may provide a new perspective on fractals and their chaotic nature, helping to solve problems involving chaotic phenomena.

INTRODUCTION

Chaos Theory

Chaos Theory attempts to explain completely unpredictable systems such as the stock market, weather, turbulence, and human organs including the heart, lungs, and brain (1). Chaos theory has numerous principles that have varied as more information about chaos was discovered, but there are six significant principles that effectively explain the fundamentals of Chaos Theory. The first principle is the phenomenon of slight changes in the initial state leading to drastic changes in the final state. The Butterfly Effect, which is the assertion that a butterfly flapping its wings in one place guarantees the occurrence of a hurricane in a place across the world, is the most used example to explain this phenomenon (2). While there are 5 other principles, this study and the rest of the principles heavily rely on this first principle.

The second principle is unpredictability: the impossibility of knowing how to control the output of a system as it is impossible to know all the initial parameters of the system. Thus, it becomes entirely possible for this output to exhibit chaotic characteristics even if a slight change is made in the initial conditions (2).

The third principle is that chaos connects order and disorder. Order comes from an established operation for which, when an input is provided to the operation, the exact output can be confidently paired with the input. On the other hand, disorder arises when a slight alteration is made to a known input-output pairing. Namely, when that input is slightly altered, it is impossible to use the established operation to determine how the corresponding output will change. The entity that is connecting this order and disorder to enable the system to continue functioning is chaos (2).

The fourth principle is mixing. Given a system with an established operation, two similar inputs can produce very different outputs, similar to a group of balloons ending up in very different places after being released from the same location (2).

The fifth principle is the fractal, or the geometric representation of chaos and one of the most prominent phenomena that appears in chaos. Fractals are infinite and intricate figures that are self-repeating at every scale, and are present at every scale in the universe, whether it be entire landscapes, biological structures, or leaves on a plant (3). Therefore, the Mandelbrot Set, one of the world's most popular models of fractals, was investigated to gain insight into the chaos of fractal nature in our world.

The sixth principle is feedback. Specifically, chaos can be amplified when there is a response to the chaos itself (3). With fractals, feedback can be understood as the process of a simple operation (Mandelbrot function) being carried out on data (points in the complex plane) and then feeding the output back into the operation.

While these six principles are the core of Chaos Theory, it is important to understand that the phenomenon of slight changes in the initial state leading to drastic changes in the final state of a system is the foundational characteristic of chaos and what much of the work in this study relies on.

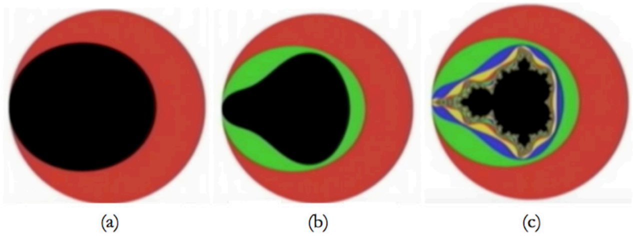

Figure 1. Stages of Mandelbrot Set iterations. a) The red and black regions combine to make up a circle with radius 2, centered at the origin of the complex plane. The red region indicates all complex numbers excluded during the first iteration, while the black region indicates all points that are in the Mandelbrot Set after the first iteration. **b)** The green region indicates all complex numbers excluded during the second iteration. The black region indicates all points that could possibly be in the Mandelbrot Set after the second iteration. **c)** The Mandelbrot Set formed by infinitely removing regions that prove to have an unbounded path after each iteration.

Mandelbrot Set

The Mandelbrot Set contains the set of complex numbers 'c' for which the function $F(z) = z^2 + c$ remains bounded for the orbit of 0, the path a function takes when iterated over 0 (**Figure 1**). The main cardioid is the main heart-shaped region of the Mandelbrot Fractal. This part of the Mandelbrot Fractal is in the section between -0.75 and 0.25 on the real axis and between approximately -0.637 and 0.637 on the imaginary axis. The approximate circles surrounding and attached to the cardioid are called primary bulbs. Any of the approximate circles not directly attached to the cardioid are simply called bulbs. For each bulb, all the points inside it approach a cycle of period n (4). Therefore, each bulb is assigned a period n. Additionally, "Minibrots" are defined as smaller figures similar in shape to the Mandelbrot itself. Minibrots are found when zooming into certain regions of the Mandelbrot Fractal (**Figure 2**) and consist of a cardioid and bulbs themselves. In other words, Minibrots are self-similar cardioid-bulb pairings emanating from all bulbs around the Mandelbrot Fractal.

Significant Points

The Mandelbrot Set contains an infinite number of points for which $z = 0$ orbits with a finite period when iterated over the function $F(z) = z^2 + c$, and every one of these points resides inside a cardioid and bulb. Furthermore, these points are denoted "center points" due to their location at the approximate center of the cardioids and bulbs (**Figure 3**).

The point where the largest primary bulb meets the main cardioid of the Mandelbrot Set (**Figure 3**) signifies a period doubling; the transition from points inside the cardioid to points inside the largest primary bulb parallels the transition from approaching a period of 1 to a period of 2. As this point is a representation of bifurcation, or the division of something into two branches, they are referred to as "bifurcation points" in this study (5). Specifically, the main cardioid's bifurcation point resides at -0.75 (**Figure 3**, red marker). The period doubling characteristic of bifurcation points leads to the fact that the derivative of the Mandelbrot function where 'c'

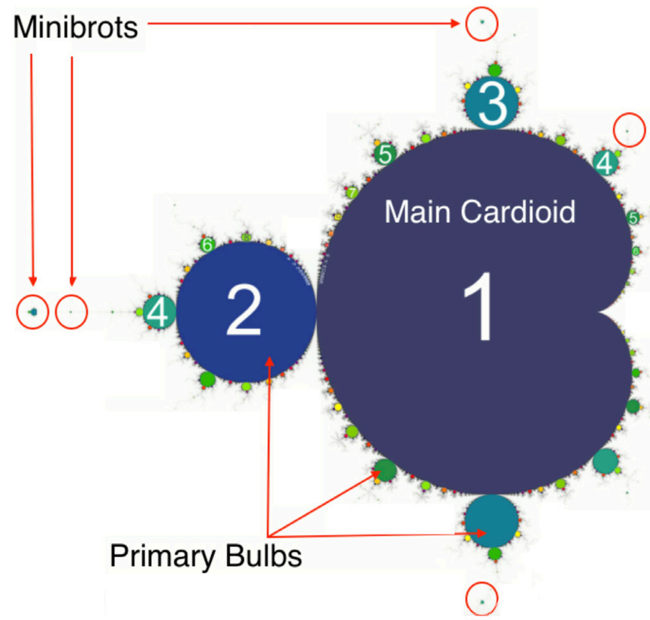

Figure 2. Mandelbrot Set components. The main cardioid and three primary bulbs are labeled. Four Minibrots are emphasized with red circles. The numbers indicate the period of each bulb.

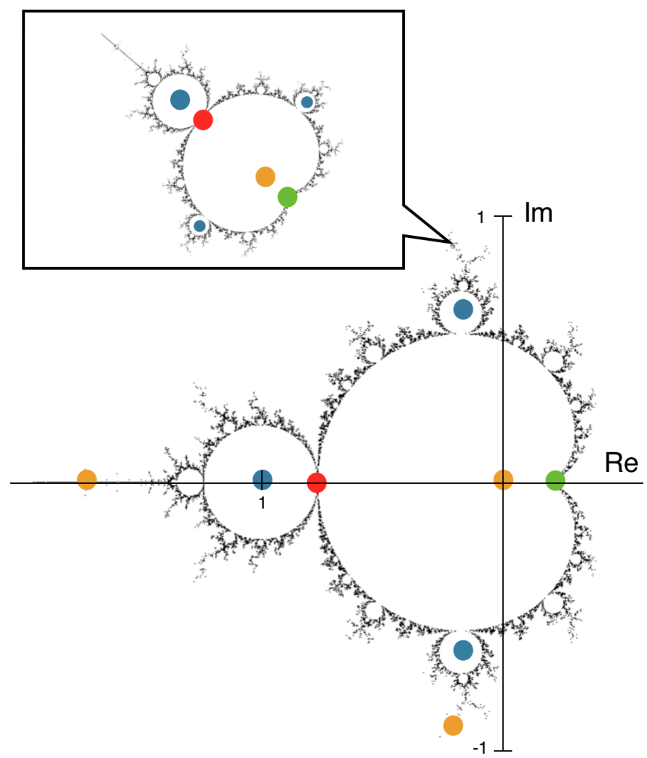

Figure 3. Significant points. Examples of center points for bulbs are shown in blue. The orange point inside the main cardioid is the center point for the main cardioid. Other orange points represent the center points for cardioids of Minibrots. Green points represent saddle points and red points represent bifurcation points. One Minibrot is enlarged to show all four of these points on a specific Minibrot.

equals the bifurcation point is greater than 1 for any input. This is because the rate of change of the approximate period increases at a factor of 2 at bifurcation points and other points along the real axis.

The point 0.25 on the Mandelbrot Set (**Figure 3**, green marker) carries significance for the opposite reason as that for the bifurcation point; this point is where, rather than the period doubling, the period simply ceases to exist. Due to the Mandelbrot Set resembling a saddle at this point, it will be referred to as the "saddle point" in this study. Contrasting bifurcation points, the derivative of the Mandelbrot function where 'c' equals the saddle point is equal to 1 for any input.

Bifurcation points and saddle points provide helpful insight into the nature of the structures they are a part of. Specifically, while bifurcation points are part of both the largest primary bulb and the cardioid, saddle points are only part of the cardioid. These properties can provide important information when locating Minibrots. Namely, once a set of center points is discovered, one can determine whether each center point resides in a bulb or cardioid by calculating whether the derivative of the Mandelbrot function where 'c' equals the center point is equal to 1 for any input. If so, then that center point is not associated with a bulb but with a cardioid, and thus, a Minibrot.

Entropy

The Kolmogorov entropy, an important measure of the degree of chaos in systems such as fractals, gives the average rate of information loss about a position of the phase point on the attractor (6). In this case, the phase point is any given point in the complex plane and the attractor is a set of values toward which a system tends to approach given many starting conditions of the system. In the Mandelbrot Set, this attribute converts nicely to the number of Minibrots in a specified region of the fractal. Specifically, the higher number of Minibrots in the region, the larger range of possible periods of inputs for which 'c' equals each point in the region; thus, this larger range is associated with a loss of information about the period. Furthermore, because the Kolmogorov entropy measures chaos, a higher number of Minibrots in a region is associated with higher entropy, which in turn is associated with more chaos.

The overlap of period states creates a region which could contain Minibrots that could have multiple possible periods, creating unpredictability and chaos (**Figure 4**). A higher level of Minibrots incidence in these regions would support these overlapping regions display a more intense chaotic nature.

RESULTS

In this study, the primary goal was to predict and explain chaotic behavior in the Mandelbrot Set. This was approached by repeatedly iterating the Mandelbrot Set's characteristic function, using an elliptical model to characterize the results from the iterations, and applying a logistical regression test to these data. The regression was employed to engender statistically significant parameters of a function that predicts the layout of the Mandelbrot Set and insight into relationships between chaos intensity and fractal locations. We hypothesized that repeatedly iterating the Mandelbrot Set's characteristic function would give rise to a more intricate layout of the fractal and elliptical models that predict and highlight "hotspots" of chaos through their overlaps. While a novel method of discovering miniature versions of the Mandelbrot fractal was discovered and a statistically significant transformation function was developed, overlaps of the elliptical models were not supported to exhibit higher levels of chaos.

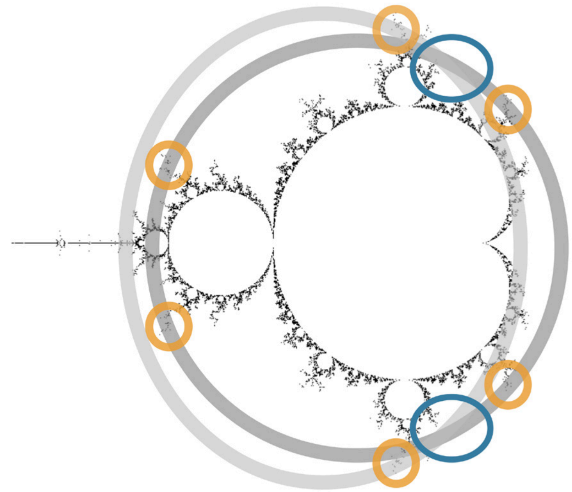

Figure 4. Examples of period states and overlaps. The two gray ellipses are the states that traverse the Mandelbrot Set, with a certain thickness due to the error margin, with intersections with Minibrots circled in orange and the overlap of states circled in blue.

Solving the Mandelbrot iteration function resulted in both center points of cardioids and bulbs that may be attached to cardioids (**Figure 5**). The Python program developed to fit an ellipse to each cardioid center set resulted in five diagrams (**Figure 6**, panels i–v). For each plot, the parameters that describe the ellipse are also given. Each parameter is relative to the period that the curve is being fit upon. To develop a transformation of a period that results in a function mapping

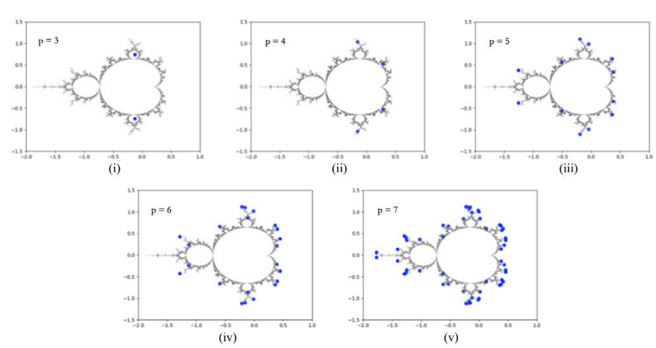

Figure 5. Calculated center points. Plot data of center points for Mandelbrot cardioids and bulbs of a certain period. Periods 3 to 7 correspond with panels (**a**) to (**e**) respectively.

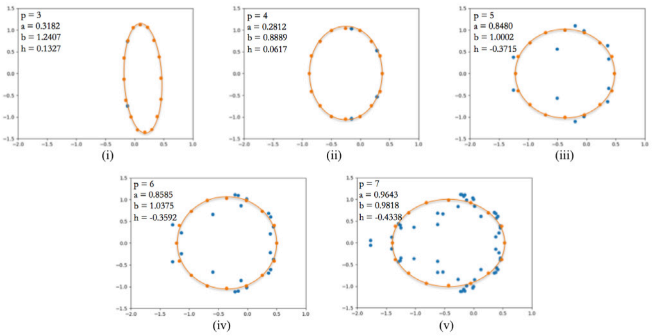

Figure 6. Calculated period states. Best fit ellipse for set of center points (blue points) for each period. Each ellipse is associated with a, b, and h values that are derived from the equation (III). Orange points are example points on the calculated ellipses spaced at equal intervals.

Table 1. Minibrot incidence and differences by period state.

Period of First State (P1)	Period of Second State (P2)	P1 Minibrot Incidence	P2 Minibrot Incidence	Difference Between Average and P1 Minibrot Incidence	Difference Between Average and P2 Minibrot Incidence
4	5	1	2	0.1100936	0.0735924
4	5	1	2	0.0677731	0.0735924
4	5	1	1	0.0677731	0.1100924
4	5	1	1	0.0677731	0.1100924
4	6	1	2	0.0677731	0.0123936
4	6	1	2	0.0677731	0.0123936
4	6	1	4	0.0677731	0.0748936
4	6	1	4	0.0677731	0.0748936
4	7	1	5	0.0677731	0.0293101
4	7	1	5	0.0677731	0.0293101
4	7	1	7	0.0677731	0.0605601
4	7	1	7	0.0677731	0.0605601
5	6	1	2	0.1100924	0.0123936
5	6	1	2	0.1100924	0.0123936
5	7	2	0	0.0735924	-0.0488150
5	7	1	0	0.1100924	-0.0488150
5	7	1	4	0.1100924	0.0136851
5	7	1	4	0.1100924	0.0136851
6	7	2	2	0.0123936	-0.0175650
6	7	2	1	0.0123936	-0.0331900
6	7	2	4	0.0123936	0.0136851
6	7	2	4	0.0123936	0.0136851

the best fit curve for that period, a relation must exist between the period and the parameters (**Figures 7-9**).

The linear regression test used to generate functions for the parameters of an elliptical curve function resulted in the following equations:

$$a = 1.137 * \ln(p) - 1.175 \quad (1)$$

$$b = -0.6400 * \ln(p) - 2.175 \quad (2)$$

$$c = -0.7530 * \ln(p) - 0.9812 \quad (3)$$

Embedded into the general elliptical curve equation symmetric about the real axis ($(x - c)^2/a^2 - y^2/b^2$), the parameters give rise to final transformation model with a relationship with the period:

$$T(p) = \frac{(x + (0.7530 * \ln(p) + 0.9812))^2}{(1.137 * \ln(p) - 1.175)^2} + \frac{y^2}{(0.6400 * \ln(p) + 2.175)^2} \quad (4)$$

The resulting incidence of Minibrots inside of overlapping period states and difference between the average Minibrot incidence and the actual Minibrot incidence of each period within each ellipse-pairing overlap show an insignificant change (**Table 1**). The various calculated ellipses for each period imposed on the same plane show the similarities for ellipses of higher periods (**Figure 10**).

DISCUSSION

The current study resulted in the development of a novel method that traverses the Mandelbrot Set to locate all Minibrots throughout the Mandelbrot Fractal. In addition, a significant transformation model was developed to predict the positions of all Minibrots of a certain period altogether. However, when ellipses were widened and overlaps of these period states were analyzed, there was no significant difference between the average Minibrot incidence and the incidence of Minibrots within the overlaps. The difference between the period state overlaps' Minibrot incidence and the average Minibrot incidence is almost negligible, with some overlaps even consisting of less Minibrots than the average number of Minibrots in a similarly sized region (**Table 1**). This result did not align with the initial intuition. Namely, since an overlap of states indicates that the main cardioid of a Minibrot found in that region does not have a predetermined period, its period has as many possibilities as the number of states that have overlapped to create that region and should produce

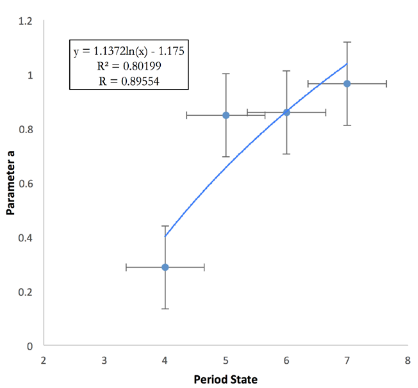

Figure 7. A parameter. Regression analysis results with 95% confidence intervals for a parameter of the transformation function. ($R = 0.896 > R\_crit = 0.811$; $p < 0.05$, $df = 4$)

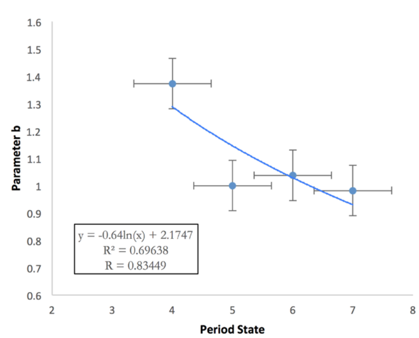

Figure 8. B parameter. Regression analysis results with 95% confidence intervals for b parameter of the transformation function. ($R = 0.834 > R\_crit = 0.811$; $p < 0.05$, $df = 4$)

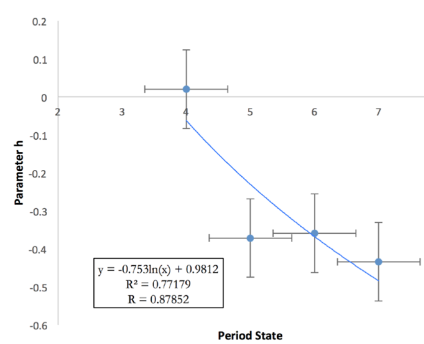

Figure 9. H parameter. Regression analysis results with 95% confidence intervals for h parameter of the transformation function. ($R = 0.879 > R\_crit = 0.811$; $p < 0.05$, $df = 4$)

higher Minibrot incidence. Therefore, instead of discovering areas associated with elevated chaos, analysis of the overlapping period states revealed that these areas exhibit as much chaos as the rest of the Mandelbrot Fractal. In fact, this result inevitably inspires the possibility that chaos (or entropy) is evenly "distributed" among the fractal space. Despite the negative result, taken together, this study did partially predict and explain chaos in the Mandelbrot Set. Looking forward, the newfound possibility of uniform chaotic behavior throughout the Mandelbrot Fractal provides numerous interesting new avenues of research if a substantial relationship between the specific Mandelbrot Fractal and the fractal nature of the universe can be established.

Based on our results, there are three major recommendations to improve future directions from this study. The first would be to use a larger sample size with not only the period state-parings for overlaps, but also with the Minibrots used to calculate the average Minibrot incidence in a certain area. By including Minibrots for which the center points cycle with higher periods, it would be possible to increase the sample size and use a similar strategy as used in this study to uncover a significant difference between the period state overlaps' Minibrot incidence and the average Minibrot incidence. The second direction to further this study would be to model the states using a figure more versatile than an ellipse to capture slight variations in the state's curve. This would then allow for less of an error margin as the state would prove a more accurate method of locating Minibrots in the Mandelbrot Set. Finally, the third direction would be to introduce more diversity by applying the same study to various fractals. This would potentially provide more insight into the purpose of chaos in dynamical systems which could then be applied to the following areas.

The Mandelbrot Fractal and Chaos Theory in general are at the core of how the universe functions (7). Furthermore, numerous significant fields, such as biology, physics, chemistry, cosmology, meteorology, and even the stock market, have been shown to follow fractal laws and exhibit chaotic behavior (8). This study provides valuable insight into three inherent mechanisms of the human body: the heart, lungs, and brain. The rate of the human heartbeat, the firing of neural clusters and the network of arteries, lung vessels, and neurons are all both chaotic in nature and can be described using fractal-based models. Intersections of certain muscles, which are essential to human biology and resilience, very closely resemble the regions of overlapping states in this study (9). If the future directions outlined above were proven more successful in confirming the entire hypothesis of this study, it may be possible to support that cardiac malfunctions, lung alveoli blockages, and neuron damage cases are associated with overlapping period state regions and can be efficiently reduced and predicted in the future using correspondences between the Mandelbrot Set and human organs (9). Therefore, the intersections of period states from this study could prove useful for diagnosis and therapeutic treatment for various diseases, development of artificial intelligence, and overall advancement of efficient technology in the future.

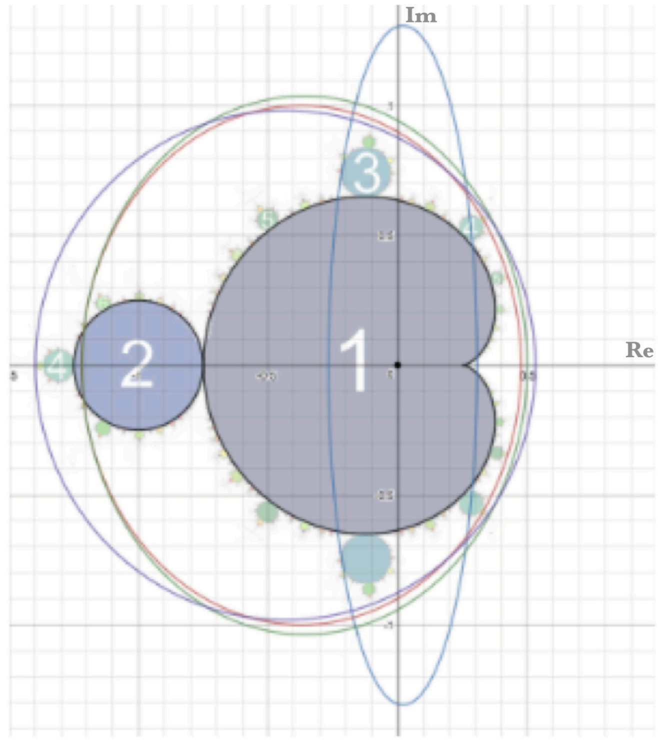

Figure 10. Calculated ellipses together. Various resulting period states, calculated using the transformation model. Figure created using Desmos.

METHODS

Phase 1: Locating Minibrots

To find the points 'c' for which *F(0)* cycles exactly with period n, it is necessary to find the solution to the equation value, that, when plugged into the n-times-iterated Mandelbrot function, equals itself. Therefore, for varying values of n, the equation $F^n(c) = c$ was solved using a program developed in Python (3.6.3). Since the solutions to this equation are in both cardioids and bulbs, it is required to find only those located in cardioids to locate the Minibrots containing those cardioids. As explained in the introduction, the equation $d/dc(F^n(c) = 1)$ was tested for each center point to determine whether each center point corresponded to a bulb or cardioid. After eliminating the solutions that correspond to center points in bulbs, the set of points that correspond to each Minibrot was established.

Phase 2: Constructing Period States

The set of solutions to $F^n(c) = c$ becomes very tedious to calculate after the first few periods. Therefore, as the portion of the Mandelbrot Fractal on the right side of the imaginary axis represents a nearly elliptical figure, and the majority of Minibrots reside on the perimeter of this "ellipse," ellipses are an ideal model for predicting the locations of all Minibrots in a certain period state. A Python program was developed to compute the best fit elliptical regression curve for all Minibrots in a given state. This was done by applying the OpenCV fitEllipse function to all the center points of the Minibrots in the respective state. After gathering the equations for these ellipses for various states, a linear regression test was used to generate functions for the parameters of an elliptical curve function with an error margin that outlines a regional state where the Minibrot could possibly be located. Note that k = 0 due to fractal symmetry. The parameter equations were then plugged into the general equation for an ellipse symmetrical over the real axis to develop the final transformation model. Finally, error margins for each ellipse were set equal to the average uncertainty in each parameter's regressions.

Phase 3: Testing Overlaps of Period States

The ellipses calculated in phase 2 combined with their error margins are referred to as period states in this study. The overlap of period states (**Figure 5**) creates a region that could contain a significantly higher number of Minibrots and thus more chaos. To determine whether overlaps truly do exhibit higher levels of chaos, each overlap was analyzed by calculating the incidence of Minibrots that it enclosed. After executing this for multiple pairs of period states that intersect, it was determined whether there is a significantly higher density of Minibrots in these overlapping regions than the average Minibrot incidence density throughout the entire Mandelbrot Set (determined by averaging the number of Minibrots inside multiple areas across the perimeter of the Mandelbrot Set roughly equal in size to the overlaps of period states).

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