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Article

Fractals: Exploring Mandelbrot Coordinates and qualitative characteristics of the corresponding Julia Set

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SUMMARY

The Mandelbrot Set and Julia Sets are fractals that are generated employing the cyclic equation $z_{n+1} = z_n^2 + C$ an infinite number of times, where z_n and C are complex numbers. For the Mandelbrot set, the value of z_n is static and the equation projects points that do not diverge to infinity for all possible values of C derived from coordinates on the complex plane. For Julia Sets, every value of C remains static while every value for z_n that does not diverge to infinity is projected. Because the standard, 2D Mandelbrot set assumes the value of $z_n = 0$, there is only one primary depiction of the Mandelbrot Set. For Julia Sets, the value of C is not assumed, and therefore, there are an infinite number of Julia Sets. In this study, the effect of changing the coordinate from which the value of C is derived in the complex plane for the equation $z_{n+1} = z_n^2 + C$ on the qualitative characteristics of the coordinate's corresponding, standard 2D Julia Set in relation to the standard, 2D Mandelbrot Set is observed — specifically, Julia Sets derived from points outside the Mandelbrot Set, in the main cardioid of the Mandelbrot Set, in the primary bulbs of the Mandelbrot Set, along the edges of the Mandelbrot set, along the real and imaginary axes.

INTRODUCTION

A fractal is a geometric image or object that is repeated in a self-similar fashion across different scales of magnification (1, 2). As such, if we magnified such an image indefinitely, a pattern would appear to repeat an infinite number of times (3). This phenomenon can best be understood using the Koch snowflake **(Figure 1)**.

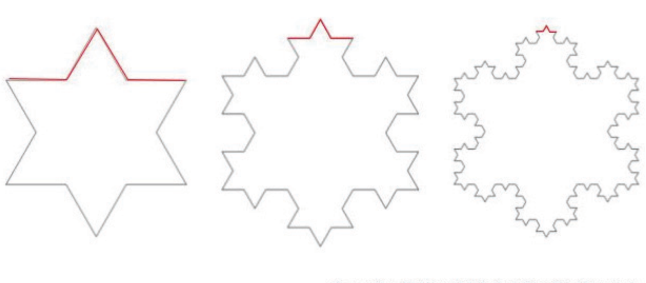


Figure 1: The Koch snowflake at different levels of iteration. Each level of "magnification" is usually referred to as an iteration. Images generated from Wolfram Alpha's Koch Snowflake Generator.

In the Koch Snowflake, four consecutive sides form a "peak" (visible in red in the first three iterations). In the first image of the Koch Snowflake, the image has 12 sides, in the second, 48 sides, and in the third, 192 sides. From the first few terms of the Koch Snowflake, we can deduce that the number of sides in the Koch Snowflake follows the equation 3*4^n (when n = the level of iteration where n > 0).

However, not all fractals are derived using a function from repetitive patterns within their first iterations. Such fractals – which include the Mandelbrot and Julia Sets – are constructed using the previous iteration, not necessarily an equation. This results in the cyclical function from which the Mandelbrot and Julia Sets are derived.

Similar to the Koch Snowflake, a repeating pattern appears upon the magnification of the Mandelbrot Set. However, unlike in the Koch snowflake where a specific part of the whole image (the "peak") is magnified, the whole Mandelbrot Set appears to repeat in the form of "Mini-Mandelbrots" in various parts of the Mandelbrot Set **(Figure 2)** (4).

From this, we see one example of how the Mandelbrot Set exhibits self-similarity across numerous scales of magnification. Such occurrences of self-similarity make the Mandelbrot Set a fractal. Similarly, Julia Sets are also fractals. However, depending on which point derives the value of z_n, the exact characteristic that exhibits self-similarity differs.

One key feature of both the Mandelbrot and Julia Sets is convergence. Convergence is an occurrence when, in mathematics, the value of an equation remains within limits at any iteration (5). The points included in cyclical functions such as those of both the Mandelbrot and Julia Sets are examples of points that exhibit the properties of convergence, where the limit is infinity (6). The Mandelbrot and Julia Sets only

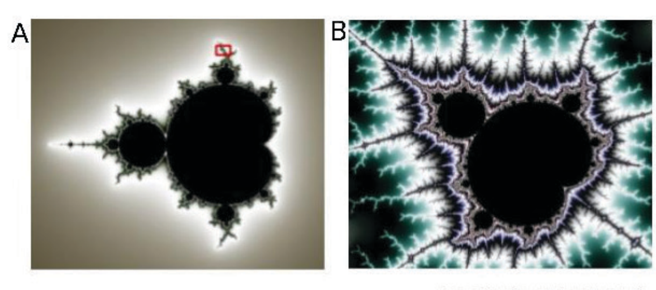


Figure 2: The Mandelbrot Set, A, and a "Mini-Mandelbrot", B. The "Mini-Mandelbrot" is the result of zooming into the space within the red box on the Mandelbrot Set.

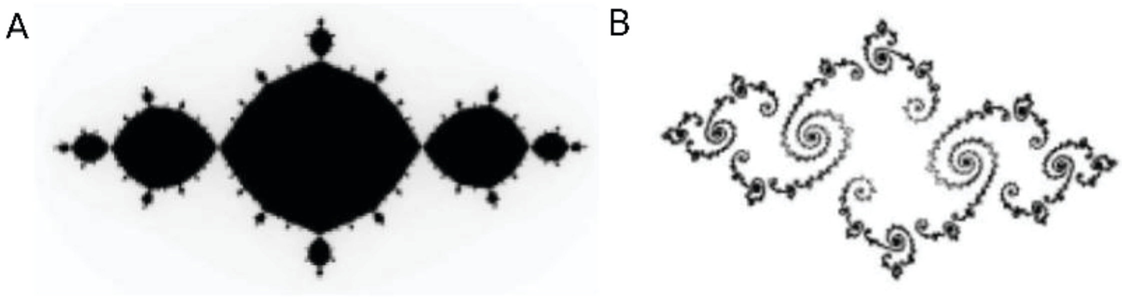


Figure 3: Filled (A) and Cantor (B) Julia Sets. The filled Julia Set (A) is derived from when C = -0.991 - 0.005i. The Cantor Julia Set (B) is derived from when C = -0.767 + 0.171i.

include points that converge to a value less than infinity (in other words, not diverging to infinity).

In 1919, both Gaston Julia and Pierre Fatou discovered that a Julia Set must be either connected or a set of an infinite number of points that do not connect (7). As such, there is no value of C in which there are 3, 73, or 785 pieces in a Julia Set (8). Either the Julia set must be connected or it must be disconnected into an infinite number of points. This phenomenon, in which there can only be two options/possibilities with no in-between, is called dichotomy. Mathematicians call connected Julia Sets "filled" Julia Sets and those that disconnected Sets as "cantor" Julia Sets **(Figure 3)**.

In this study, Julia Sets were differentiated by whether they were filled or cantor sets as well as by their number of junctions (number of largest proximal bulbs adjacent to primary bulb of Julia Set). The relationship between the period of the Julia Sets and the Mandelbrot Set was also a key focus.

The period of a function is the number of iterations the function takes the resulting values to repeat in a pattern-like fashion (9). For example, in the cyclic equation for the Mandelbrot Set/Julia Sets $z_{n+1} = z_n^{\ 2} + C$, when the value of C = 0 and $Z_1 = 0$, the value of nx (level of iteration) is static. Therefore, at every iteration of the function the same number repeats and the orbit, or the number of values in the iterations which repeat in a pattern of n_x is 1 (the value of n_x stays constant), so this particular function has a period of 1. When C = 1 there appears to be no pattern, which is referred to as chaos. If we were to calculate n∞, the value of the iteration would also be infinity, since the value of z_{n+1} has no limit and is not bound (8). Therefore, this function does not have a period as it does not ever repeat in a pattern. The period of the equation when C = 1 presents a special case. When C = -1, the value of nx oscillates between 0 and -1. Because this equation causes the orbit to switch between two values, it has a period of 2 (10). Previous literature has identified that as a point enters into a primary bulb of a Julia Set, the number of regions directly connected to the first set of junctions of the Julia Set on either vertex is equal to the period of the Bulb the coordinates used to create the Julia Set is located within the Mandelbrot Set subtracted by one.

In this study, it was hypothesized that as the point from which the Julia Set is derived becomes closer to the center of the Mandelbrot Set (defined as the origin of the complex plane), the more complex in pattern that point's Julia Set will become (measured qualitatively).

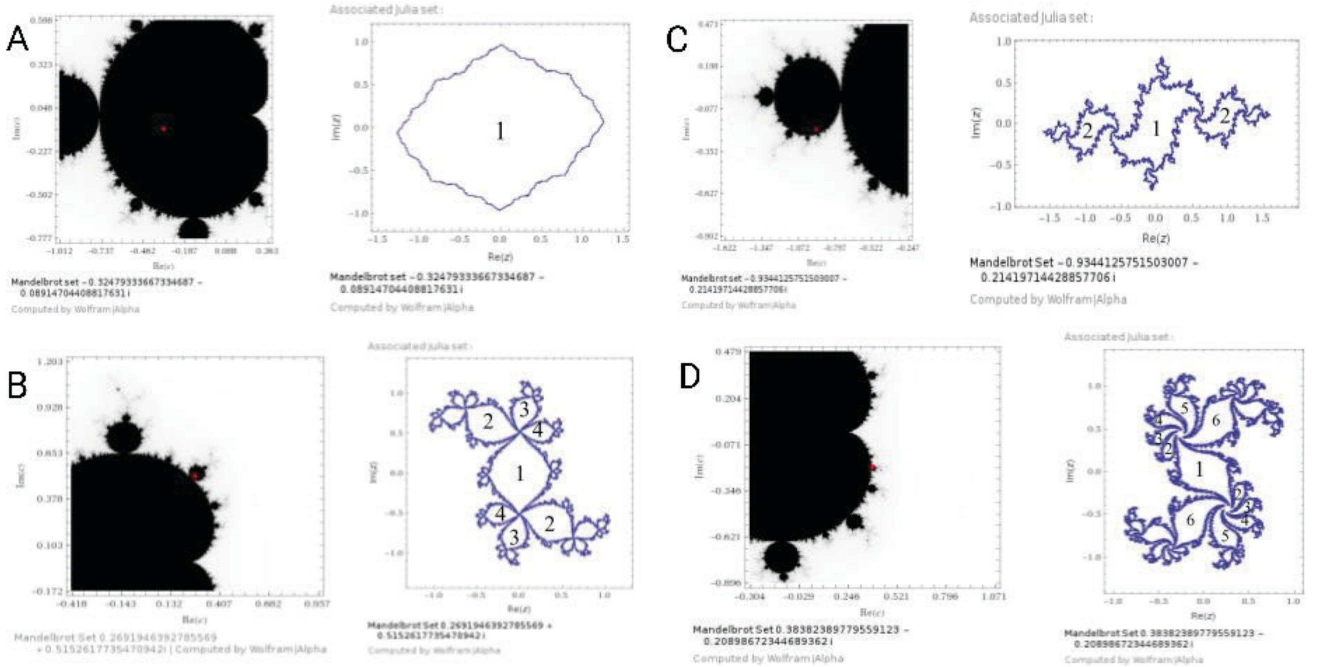
RESULTS

Upon analysis of the generated Julia Sets, four primary trends and three additional findings were found between a coordinate's location on the complex plane and Mandelbrot Set and the characteristics of that point's corresponding Julia Set.

The complexity of a Julia Set was defined using the number of periods of the Julia Set, the number of observable junctures, and the type of Julia Set (Cantor vs. Filled) by qualitatively analyzing the images of Julia Sets produced from different locations on the complex plane in relation to the Mandelbrot Set.

Correlations between the Location of a Coordinate in the Mandelbrot Set and the Period of the Corresponding Julia Set

As a coordinate in the complex plane that is within the Mandelbrot Set enters a bulb — whether the bulb is a primary, secondary, tertiary, or another type of bulb — of the Mandelbrot Set, the period of the corresponding Julia Set becomes equal to the period of the bulb in which the coordinate is located. Therefore, if a coordinate lies in a part of the Mandelbrot Set with a period of 1 (the main cardioid), the Julia Set was identified to also have a period of 1. Because of the correlation between periods and junctions, junctions were used to identify periodal relationships **(Figure 4)**.
Notice how the point from which the point in 4C, relative to the Mandelbrot Set, is in a primary bulb, which would have a period of 2. The point's corresponding Julia Set has two primary junctions (labeled "2"), making the number of primary junctions of a Julia Set an indicator of the bulb from which it was derived from, and what the period of that bulb is.



(Images from Wolfram Alpha's Mandelbrot Set Generator. Image of Julia Set is edited for clarity)

Figure 4: Sample of Julia Sets within the Mandelbrot Set and the relative location of the point that they are derived from on the Mandelbrot Set. Julia Sets are edited with numbers to clarify regions connected to central junctions.

Correlations between a Coordinate's Location Relative to Real Axis, Imaginary Axis, and Origin and the Point's Corresponding Julia Set

It was found that as a point on the complex plane moves along the real axis (x-axis on the complex plane) or along the imaginary axis (y-axis on the complex plane), the corresponding Julia Sets are symmetric across the origin, regardless of the location of the coordinate inside, outside, around, or in any bulb of the Mandelbrot Set **(Figure 5)**.

As values on the real axis become more positive, the corresponding Julia Set vertically stretches, while maintaining radial symmetry. On the real axis, negative values assign the point's corresponding Julia Set an orientation along the line y = x, while positive values assign the point's corresponding Julia Set an orientation along the line y = -2.

Correlations between the Location of a Coordinate in Relation to the Perimeter of the Mandelbrot Set and the Corresponding Julia Set's Complexity

As a point neared the inner or outer perimeter of the Mandelbrot Set, the relative complexity of the point's corresponding Julia Set increased. No apparent difference between points inside the Mandelbrot Set vs outside the Mandelbrot Set were apparent, except that points inside the Mandelbrot Set generated filled Julia Sets and points outside the Mandelbrot Set generated Cantor Julia Sets **(Figure 6)**.

Notice how the other finding on symmetry reveals itself in the Julia Sets in **Figure 6**, while the finding on junctions only reveals itself in the closed Julia Sets, because Cantor Julia Sets, with their infinite number of points, would not have actual junctions, as no two points are connected.

Additional Findings

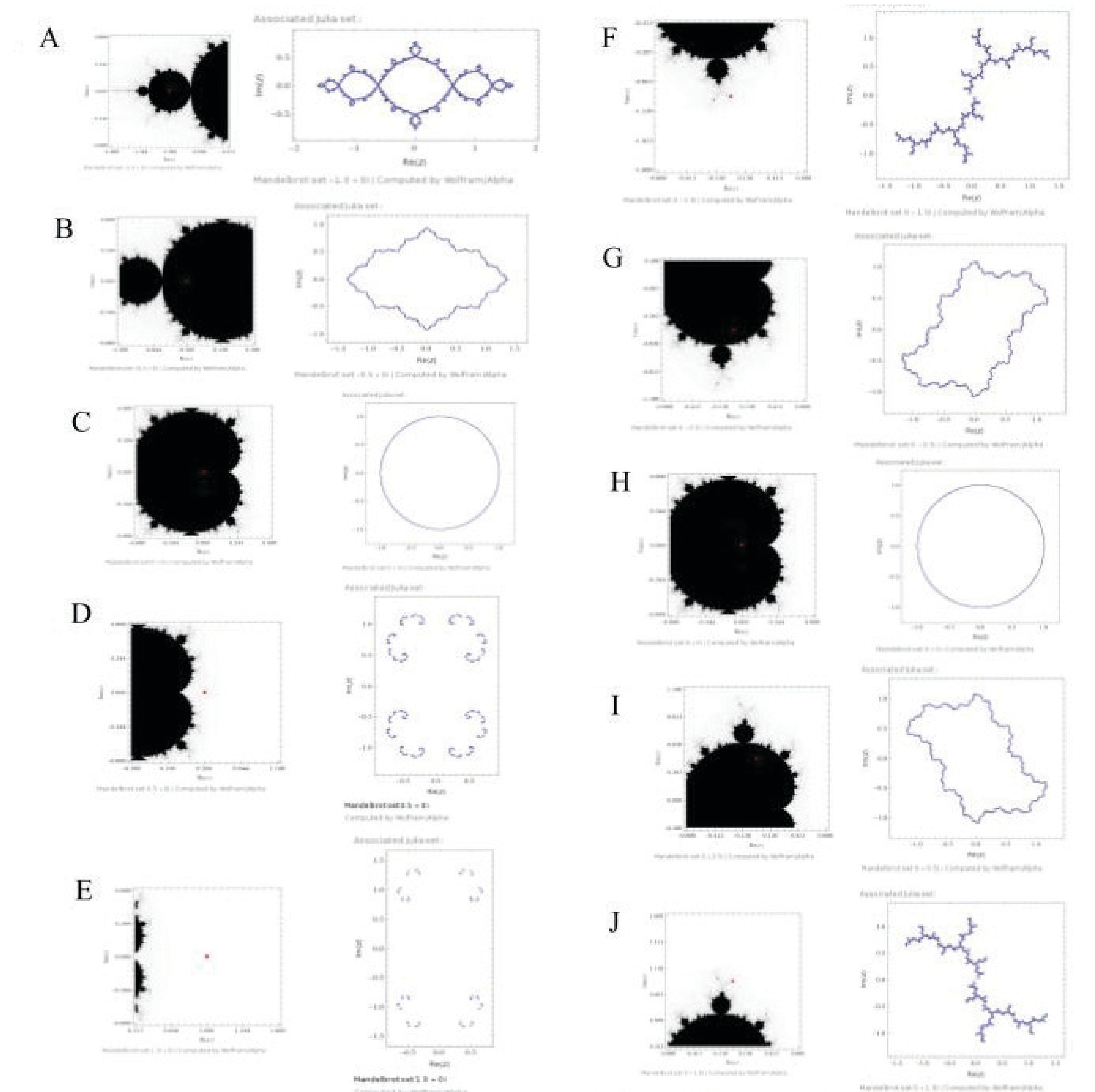
Although this study focused on how moving the coordinate of a point on the complex plane in relation to the Mandelbrot Set and the axes impact the corresponding Julia set, three additional patterns in Mandelbrot-Julia Set behavior were discovered.

We found that the location of a coordinate in relation to the Mandelbrot Set as a whole correlated with the point's corresponding Julia Set. When a point was located within the Mandelbrot Set, the point's corresponding Julia Set was a filled Julia set and when a point was located outside of the Mandelbrot Set, the corresponding Julia Set was a cantor set. Therefore, the dichotomy of a Julia Set is related to the location of the coordinate the Julia set is derived from in relation to the Mandelbrot Set **(Figure 6)**.

We also found that as a coordinate on the complex plane moves toward the origin, the Julia Set tends to a circular shape with the Julia Set corresponding to the origin of the complex plane as a unit circle (see **Figures 4 and 5**).

In addition to this, it was found that all points on the complex plane and in the Mandelbrot Set whose corresponding Julia Set is within a radius of 2 units about the origin are filled.

In other words, all points within the Mandelbrot Set are located within a radius of 2 units about the origin. Upon



(Images from Wolfram Alpha's Mandelbrot Set Generator)

Figure 5: Sample of Julia Sets as the point they are derived from moves along real and imaginary axes. A-E depict the corresponding Julia Set when the point that it is derived from moves across the real plane. F-J depict the corresponding Julia Set when the point that it is derived from moves across the imaginary plane. The coordinate's location relative to the Mandelbrot Set is also shown. Notice the symmetry across the origin for each Julia Set, regardless of location relative to the Mandelbrot Set.

crossing this boundary, all points are no longer within the Mandelbrot Set. As a result, it was also concluded that all Julia Sets that are filled have the coordinate that they are derived from located within a radius of 2 units about the origin of the complex plane.

DISCUSSION

The analysis of the images concluded that as a point moves farther away from the perimeter of the Mandelbrot set, whether the point is inside or outside of the Mandelbrot Set, the corresponding Julia Set's pattern becomes qualitatively simpler and more distributed.

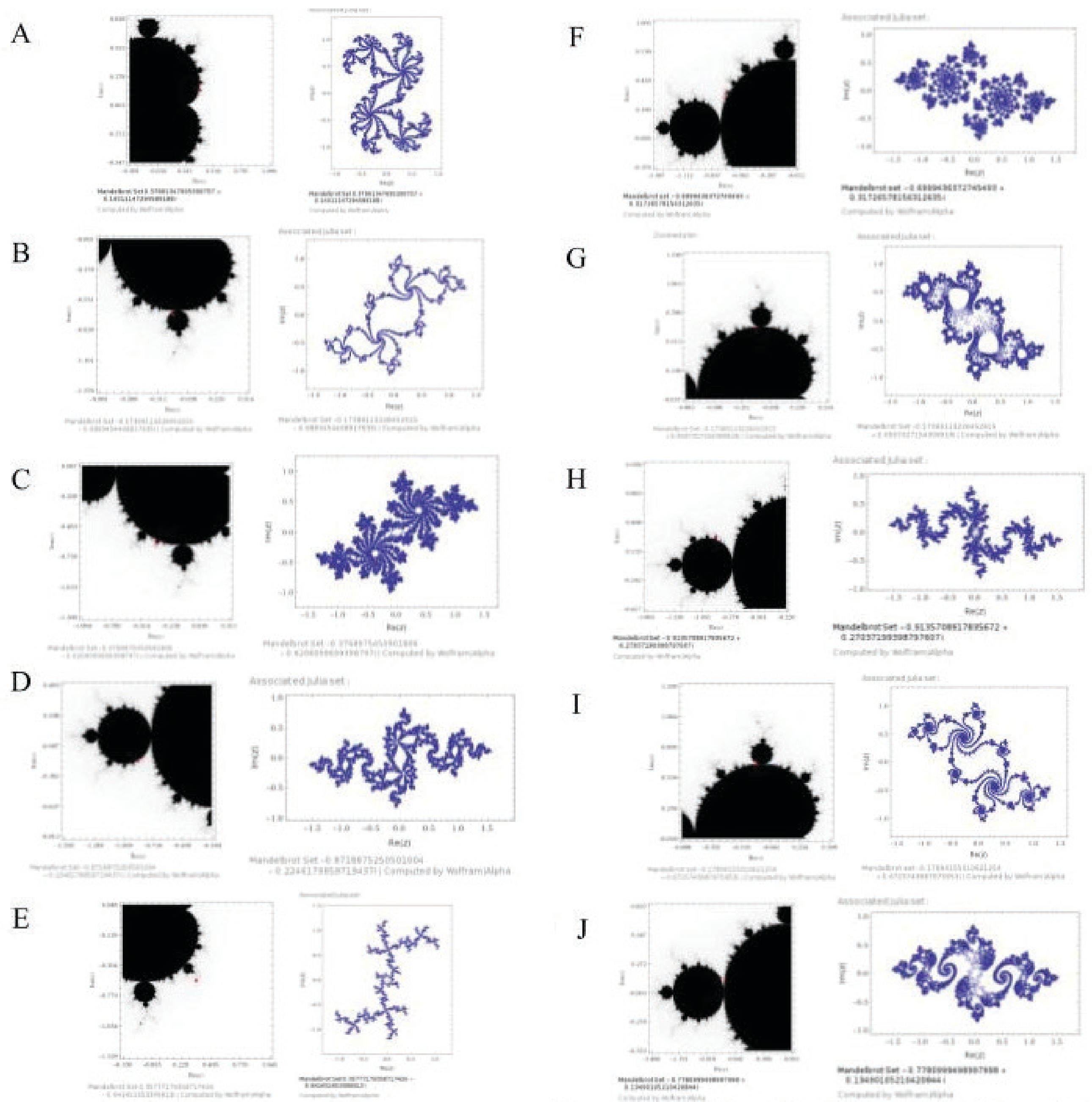


Figure 6: Sample of Julia Sets as the point they are derived from moves closer to the perimeter on the corresponding Mandelbrot Set. A-E depict the Julia Set for coordinates near the Mandelbrot Set's inner perimeter. F-J depict the Julia Set for coordinates near the Mandelbrot Set's outer perimeter. Notice how points that are relatively closer to the perimeter become dramatically more complex than those farther from the perimeter. Note how all Julia Sets generated from points within the Mandelbrot Set (A-E) are filled Julia Sets and all Julia Sets generated from points outside of the Mandelbrot Set (F-J) are cantor Julia Sets.

On the other hand, as the point moves closer to the perimeter, whether inside or outside of the Mandelbrot Set, the opposite is true: the corresponding Julia Set is more clustered and complex in design. When a point is inside of the Mandelbrot Set, a similar phenomenon can be observed. As the farther a point in the Mandelbrot Set is from the Mandelbrot Set's perimeter, the less complex it is than in comparison to a point in the Mandelbrot Set that is closer to the perimeter.

To the eyes of a viewer of two Julia Sets, one of which has derivative coordinates closer to the perimeter of the Mandelbrot Set, the Julia Set derived from the point closer to the perimeter appears to be more clustered. However, due to the Julia Set's nature of fundamental dichotomy, there is no difference in the density of the Julia sets closer to or farther from the Mandelbrot Set's perimeter. This observation can best be understood as the result of our perception of the junctions of the Julia Sets.

These findings show how the complexity of the Mandelbrot and Julia Sets are governed by relatively simple rules. The fact that the characteristics of a Julia Set can be predicted with such a level of accuracy suggests that the two sets are deeply interconnected, although the Mandelbrot Set and Julia Set represent different values in the equation $z_{n+1} = z_n^2 + C$. These findings reveal one of the most complicated things around us — fractals — are governed by simple rules and patterns that can be identified and learned from. Although these findings further indicated preexisting knowledge of trends in Julia Set behavior, such correlations and patterns open the door to the possibility of defining, predicting, creating, and analyzing the many fractals in the real world.

MATERIALS AND METHODS

In order to conduct this analysis, a manipulative Julia Set Generator was sourced from Mark McClure, Ph.D., Professor of Mathematics at the University of North Carolina in Asheville's open-source program. With the generator a specific point on the Mandelbrot Set could be selected with which an associated coordinate can be generated. WolframAlpha's open-source Mandelbrot Set Generator was used to offer a more accurate image of the associated Julia Set.

Using the Julia Set Generator, a point on or off the Mandelbrot Set was selected, depending on the location of the point in relation to the Mandelbrot Set. This coordinate was inputted into WolframAlpha's Mandelbrot Set Generator and both the image of the point on the zoomed Mandelbrot Set as well as the corresponding Julia Set produced from the program was copied for later analysis. This process was repeated for 111 points on the complex-plane.

General locations of coordinates used to derive Julia Sets relative to Mandelbrot Set:

- 20 coordinates far from the Mandelbrot Set perimeter, outside (5 to upper left, upper right, lower left, and lower right)
- 10 coordinates close to the Mandelbrot Set perimeter, outside (throughout)
- 10 coordinates close to the Mandelbrot Set perimeter, inside (throughout)
- 10 coordinates in the main cardioid of the Mandelbrot Set
- 10 coordinates in the largest primary bulb of the Mandelbrot Set
- 10 coordinates across other primary bulbs of the Mandelbrot Set
- 10 coordinates across other bulbs of the Mandelbrot Set (not primary)
- 10 coordinates across other projections of the Mandelbrot Set
- 10 coordinates along X-axis (from x=-2.5 to x=2.5, by 0.5 units, excluding when x=0)
- 10 coordinates along Y-axis (from y=-2.5 to y=2.5, by 0.5 units, excluding when y=0)
- 1 coordinate at the origin

For this study, the focus was on points outside of the Mandelbrot Set, outside the Mandelbrot Set but close to the perimeter of the Mandelbrot Set, inside the Mandelbrot Set but close to the perimeter in various locations, inside of the Main Cardioid of the Mandelbrot Set, inside of the largest Primary Bulb of the Mandelbrot Set, inside other Primary Bulbs of the Mandelbrot Set, in Secondary and Tertiary Bulbs of the Mandelbrot Set, in other projections in the Mandelbrot Set, on the origin of the complex plane, and along various points on the real and imaginary axes. The complexity of a Julia Set was defined as the number of observable periods of the Julia Set, the number of observable junctures, and the type of Julia Set (Cantor vs. Filled).

To measure complexity, a comparative analysis was conducted on the images of Julia Sets produced with other Julia Sets that were derived from different locations on the complex plane in relation to the Mandelbrot Set qualitatively.

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